12.2 <u>Vectors</u>

Def: A vector in R^3 is a directed line segment
picture (in R^2)

"head" or "tip" or "terminus"

(1,1)
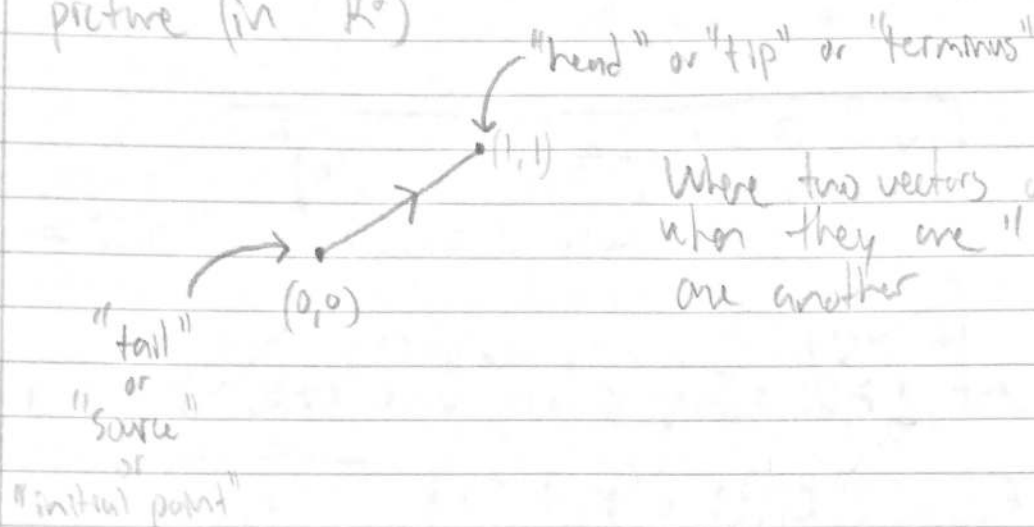
Where two vectors are equivalent
when they are "linear shifts" of
one another

(0,0)

"tail"
or
"source"
or
"initial point"

<u>Operations on Vectors</u>

1. Magnitude   (Vector $\longmapsto$ real number ≥ 0)
   $|\vec{v}| =$ length of a segment representing $\vec{v}$

(3,4)

(1,3)
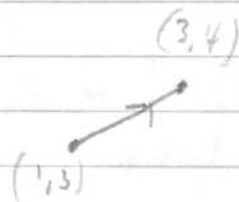
$|\vec{v}|$ length $(\vec{51})$

2. Addition   (vector + vector $\longmapsto$ vector)

$\vec{u}$
$\vec{v}$

$\vec{u}$ $\vec{v}$
$\vec{u}+\vec{v}$

"Tip to
tail"
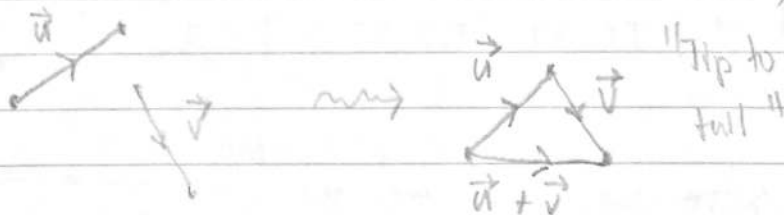
$= \sqrt{(3-1)^2 + (4-3)^2}$
$= \sqrt{4+1} = \sqrt{5}$

3. Subtraction   (Vector − Vector $\longmapsto$ Vector)

$\vec{u}$
$\vec{v}$
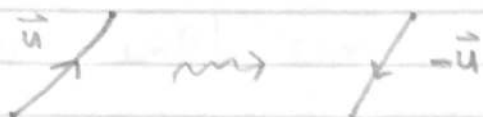
$\vec{u}-\vec{v}$ "Tip to
Tip"
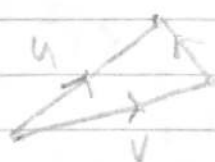
Away from the
tip you're subtracting

Ex: Zero Vector

The only vector
with magnitude zero

4. Negation   (Vector $\longmapsto$ Vector)

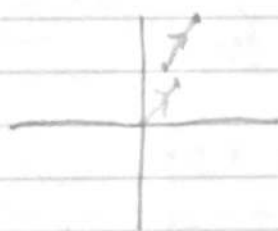 - $\vec{u}$ is obtained from $\vec{u}$ by "flipping" it

$$\vec{u} \nearrow \quad \leadsto \quad \swarrow -\vec{u}$$

real number

5. Scalar Multiplication (scalar · vector $\longmapsto$ vector)
   "stretching"

$$\vec{u} \nearrow$$

$c \geq 0$ $c > 1$ $\nearrow$ $c\vec{v} \nearrow$ longer (+)

$c < 1$ $\searrow$

$c < 0$ $c\vec{v} \nearrow$ shorter (−)
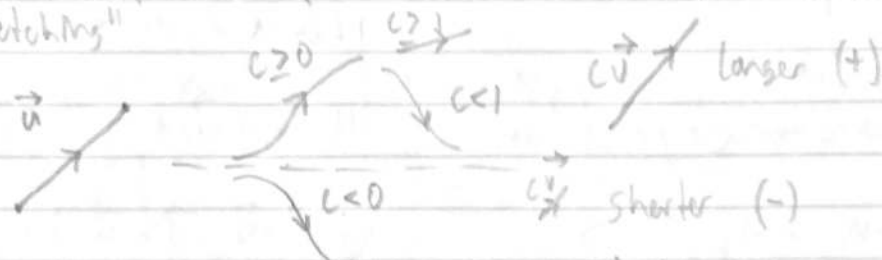
$\swarrow c\vec{v}$ Stretch/contract is determined by absolute value of c for negative

Every vector has a unique representation with tail at the origin

components of $\vec{v}$

write $\vec{v} = \langle 1, 1 \rangle$

To compute the component representation of $\vec{v}$, take a representation and compute "tip minus tail"

Ex: The vector represented by the segment subtracted from $(-3, 7)$ to $(-5, 11)$ has components

$$\vec{v} = \langle -5 - (-3), 11 - 7 \rangle = \langle -2, 4 \rangle$$

Ex: The zero vector is the vector with all components 0.

in R^3: $\vec{0} = \langle 0, 0, 0 \rangle$
in R^4: $\vec{0} = \langle 0, 0, 0, 0 \rangle$

(Below we write in 3-space, but n-space is analogous)

① Magnitude

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{u} \langle u_1, u_2, u_3 \rangle$ $c \in$ Real #'s

$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ ← Immediate from the distance formula

② Addition

$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
→ componentwise addition!

③ Subtraction

$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$
→ componentwise subtraction!

④ Negation

$-\vec{u} = \langle -u_1, -u_2, -u_3 \rangle$

⑤ Scalar Multiplication

$c\vec{u} = \langle cu_1, cu_2, cu_3 \rangle$

Thus (Properties of Vector Operations)

Let $\vec{u}, \vec{v}, \vec{w} \in R^3$ and $a, b \in$

① $(\vec{u} + \vec{v}) + \vec{w}$ or $\vec{u} + (\vec{v} + \vec{w})$ Assoc

② $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ Commutative

③ $\vec{0} + \vec{v} = \vec{v}$ Identity

④ $\vec{v} + (-\vec{v}) = \vec{0}$ Negatives

⑤ $a(b\vec{v}) = (ab)\vec{v}$

⑥ $(a+b)\vec{v} = a\vec{v} + b\vec{v}$

⑦ $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

⑧ $0\vec{v} = \vec{0}$ and $1\vec{v} = \vec{v}$

Scalar regrou

Adding two vectors only works if they belong to the same space

So $\langle 3, -7 \rangle + \langle 5, 1, 0 \rangle$ is nonsense!

Scalar Multiplication really needs scalar $\vec{u}\vec{v}$ is nonsense!

It's a really good exercise to prove the theorem for R^3

Direction

Prop: Given $\vec{u} \in \mathbb{R}^n$ and $C \in \mathbb{R}$,
$$|C\vec{u}| = |C||\vec{u}|$$

Also, $|\vec{u}| = 0$ if and only if $\vec{u} = \vec{0}$

Definition of direction: The direction of vector $\vec{v} \neq \vec{0}$ is the unit vector (ie. vector of length 1) obtained from $\vec{v}$. That is, $\frac{1}{|\vec{v}|}\vec{v}$

$\frac{1}{|\vec{v}|}\vec{v}$ is a unit vector

why? $\left|\frac{1}{|\vec{v}|}\vec{v}\right| = \left|\frac{1}{|\vec{v}|}\right||\vec{v}| = \frac{1}{|\vec{v}|}|\vec{v}| = 1$

Standard basis in $\mathbb{R}^3$:

$$\left.\begin{array}{l}\vec{i} = \langle 1,0,0\rangle \\ \vec{j} = \langle 0,1,0\rangle \\ \vec{k} = \langle 0,0,1\rangle\end{array}\right]$$ Form the standard basis for $\mathbb{R}^3$

Every vector $\vec{v} = \langle V_1, V_2, V_3\rangle$

$= \langle V_1,0,0\rangle + \langle 0,V_2,0\rangle + \langle 0,0,V_3\rangle$

$V_1\langle 1,0,0\rangle + V_2\langle 0,1,0\rangle + V_3\langle 0,0,1\rangle$

$= V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$